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Identical approximative sequence for various notions of universality

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Abstract

In this paper, we examine various notions of universality, which have already been proved generic. Our main purpose is to prove that generically they occur simultaneously with the same approximative sequence.

MSC: 30B30

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1. Introduction

Let $\Omega \subsetneqq \mathbb{C}$ be a simply connected domain and $f \in H(\Omega)$. For $\zeta \in \Omega$ we denote by $S_N(f,\zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n$ the Nth partial sum of f with center ζ.

Definition 1.1. A holomorphic function $f \in H(\Omega)$ belongs to the class $U(\Omega)$ if for every compact set $K \subset \Omega^c$ with K^c connected and for every function $h : K \longrightarrow \mathbb{C}$ continuous on K and holomorphic in K^o, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of natural

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numbers such that:

$$\sup_{\zeta\in L}\sup_{z\in K}|S_{\lambda_n}(f,\zeta)(z)-h(z)|\longrightarrow 0\,,\quad n\to+\infty$$

for every compact set $L\subset\Omega$.

The above class of functions, which are called universal Taylor series, was introduced in [20] and it was proved G_δ and dense in $H(\Omega)$ with the topology of uniform convergence on compacta. In [18] the following class was also introduced and proved G_δ and dense in $H(\Omega)$.

Definition 1.2. A holomorphic function $f\in H(\Omega)$ belongs to the class $B(\Omega)$ if there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of natural numbers such that for all compact sets $L,\widetilde{L}\subset\Omega$ we have

$$\sup_{\zeta\in L}\sup_{z\in\widetilde{L}}|S_{\mu_n}(f,\zeta)(z)-f(z)|\longrightarrow 0\,,\quad n\to+\infty.$$

Since $H(\Omega)$ is a complete metrizable space and both $U(\Omega)$ and $B(\Omega)$ were proved G_δ and dense in $H(\Omega)$, their intersection is also G_δ and dense in $H(\Omega)$. This argument is valid for various notions of universality and we obviously obtain that there exist functions which are simultaneously universal in various notions. In the past constructive proofs were considered for the existence of universal functions and in [16] a long and technical proof is given only to prove that the intersection of classes of universal functions is dense (and not G_δ dense) in $H(\Omega)$. This shows us the power of proofs using Baire's theorem. For the importance of generic results and the role of Baire's theorem in complex, harmonic or functional analysis we refer to [10,8].

In this paper we are concerned with the following question "Do they exist functions, universal in various senses, which combine these universalities harmonically i.e. with the same approximative sequence?" In [18] it was proved that it is possible to find a function $f\in U(\Omega)\cap B(\Omega)$ which realizes both approximations with the same subsequence of partial sums of Taylor expansion of the function, i.e. $\lambda_n=\mu_n$ (in Definitions 1.1 and 1.2). In view of the above result (which was generic) we study the same question for various notions of universality.

Let us recall some notation and give two definitions. Following [12,13], if $f\in H(\Omega)$, a sequence $f^{(-n)}$ is called a strict sequence of n-fold antiderivatives of f, if for every $z\in\Omega$ the following hold:

$$f^{(0)}(z)=f(z)$$

and

$$\frac{d}{dz}f^{(-n-1)}(z)=f^{(-n)}(z)\quad\text{for every } n=0,1,2,\ldots\,.$$

Definition 1.3. Let $f\in H(\Omega)$ and let $f^{(-n)}$ be a strict sequence of n-fold antiderivatives of f. We say that the sequence $f^{(-n)}$ is a strict universal sequence of n-fold antiderivatives of f if for every compact set $L\subset\Omega$ with connected complement and every function

$\varphi : L \longrightarrow \mathbb{C}$, continuous on L and holomorphic in L^o, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that:

$$\sup_{z\in L} |f^{(-\lambda_n)} - \varphi(z)| \longrightarrow 0\ , \quad n \to +\infty.$$

Now, we may consider two sequences of complex numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that:

- $a_n \neq 0$ for every $n \in \mathbb{N}$.
- Every boundary point of Ω is an accumulation point of $\{b_n\}_{n\in\mathbb{N}}$.
- For every compact set $M \subset \mathbb{C}$ there exists a natural number n_0 such that $a_n z + b_n \in \Omega$ for every $n \in \mathbb{N}$, $n \geqslant n_0$ and for every $z \in M$.

Definition 1.4. A holomorphic function $f \in H(\Omega)$ belongs to the class $V(\Omega, \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}})$, if for every compact set $K \subset \Omega^{\mathrm{c}}$ with K^{c} connected, for compact set $M \subset \mathbb{C}$ with M^{c} connected, for every choice of compact sets $L', \widetilde{L} \subset \Omega$ with connected complements, for every choice of compact sets $L, \widehat{L}, \check{L} \subset \Omega$, for every $b \in \partial\Omega$ and for every choice of continuous functions as follows

$$\begin{array}{ll} h : K \longrightarrow \mathbb{C} & h \text{ holomorphic in } K^o, \\ \omega : M \longrightarrow \mathbb{C} & \omega \text{ holomorphic in } M^o, \\ \psi : L' \longrightarrow \mathbb{C} & \psi \text{ holomorphic in } L'^o, \\ \varphi : \widetilde{L} \longrightarrow \mathbb{C} & \varphi \text{ holomorphic in } \widetilde{L}^o, \end{array}$$

there exist a sequence $\{\lambda_n\}_{n\in N}$ and a strict universal sequence of antiderivatives $f^{(-n)}$ of f such that

1. $$\sup_{\zeta\in\widehat{L}} \sup_{z\in\check{L}} |S_{\lambda_n}(f,\zeta)(z) - f(z)| \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow +\infty,$$
2. $$\sup_{\zeta\in L} \sup_{z\in K} |S_{\lambda_n}(f,\zeta)(z) - h(z)| \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow +\infty,$$
3. $$\sup_{z\in M} |f(a_{\lambda_n} z + b_{\lambda_n}) - \omega(z)| \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow +\infty,$$
   $$\text{and} \qquad |b_{\lambda_n} - b| \to 0$$
4. $$\sup_{z\in L'} |f^{(\lambda_n)}(z) - \psi(z)| \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow +\infty,$$
5. $$\sup_{z\in\widetilde{L}} |f^{(-\lambda_n)}(z) - \varphi(z)| \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow +\infty.$$

We shall prove that the class $V(\Omega, \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}})$ is non-empty. In fact our main result states that this class is G_δ and dense in $H(\Omega)$ (Theorem 2.5). Moreover at the end of the paper we give another definition (Definition 2.6) which is equivalent to Definition 1.4. The difference is that the sequence b_{λ_n} does not converge to a point of $\partial\Omega$ but it accumulates to every point of $\partial\Omega$.

In the above definition property 2 of f implies that f is a universal Taylor series. This class of functions and similar other classes, concerning also property 1, where studied in [18–20]; see also (for weaker results) [3,11,14,7]. The class of functions which satisfy property 3 of Definition 1.4 were studied in [15] (see also [2,7]). For a treatise in the fourth universality see [17,5,7,6,1,4] . Finally the fifth universality appeared in [13,14,1]. In [16] all the above universalities appear and a function is constructed with all the above properties, but as we mentioned before the approximation does not occur with the same approximative sequence. Baire's method leads us naturally and easily to simultaneous universalities with the same approximative sequence of indices.

2. The class $V(\Omega, \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}})$ is residual

Let $D_m = \{z \in \mathbb{C} : |z| \leqslant m\}$, $m = 1, 2, \ldots$ and let $\{\zeta^{(\ell)}\}_{\ell\in\mathbb{N}}$ be a sequence of points in $\partial\Omega$ which are dense in $\partial\Omega$. In addition, let $\{L_m\}_{m\in\mathbb{N}}$ be an exhausting family of compact subsets in Ω (see [21]). Because Ω is a simply connected domain, the sequence $\{L_m\}_{m\in\mathbb{N}}$ can be chosen such that L_m^c is connected. Moreover let $\{K_m\}_{m\in\mathbb{N}}$ be a sequence of compact subsets of Ω^c such that K_m^c is connected satisfying the following property: for every $K \subset \Omega^c$ compact with K^c connected, there exists $m \in \mathbb{N}$ such that $K \subset K_m$ (see [18]). We can also consider that for each m, the set $\{m' : K_{m'} = K_m\}$ is infinite. Finally, let $\{f_j\}_{j\in\mathbb{N}}$ be an enumeration of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Then, for every $m, \ell, j_1, j_2, j_3, s$ and $n \in \mathbb{N}$ we define the set $E(m, \ell, j_1, j_2, j_3, , s, n)$ as following:
A holomorphic function $g \in H(\Omega)$ belongs to $E(m, \ell, j_1, j_2, j_3, s, n)$ if it satisfies the following four properties:

1. $$\sup_{\zeta\in L_m} \sup_{z\in L_m} |S_n(g, \zeta)(z) - g(z)| < \frac{1}{s},$$
2. $$\sup_{\zeta\in L_m} \sup_{z\in K_m} |S_n(g, \zeta)(z) - f_{j_1}(z)| < \frac{1}{s},$$
3. $$\sup_{z\in D_m} |g(a_n z + b_n) - f_{j_2}(z)| < \frac{1}{s} \quad \text{and} \quad |b_n - \zeta^{(\ell)}| < \frac{1}{s},$$
4. $$\sup_{z\in L_m} |g^{(n)}(z) - f_{j_3}(z)| < \frac{1}{s}.$$

Remark. For many indices n the above set maybe empty, but for every $m \in \mathbb{N}$ there exists infinitely many indices n such that the set is not empty.

Lemma 2.1. *For every function $f \in H(\Omega)$, any function $\varphi \in H(\Omega)$ and any sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ there exists a strict universal sequence $f^{(-n)}$ of n-fold antiderivatives of f such that*:

$$f^{(-\lambda'_n)}(z) \longrightarrow \varphi(z) \quad as \quad n \to +\infty$$

locally uniformly in Ω, where $\{\lambda'_n\}_{n\in\mathbb{N}}$ is a subsequence of $\{\lambda_n\}_{n\in\mathbb{N}}$.

Proof. We only need to modify a little the proof of theorem in [12]. In this lemma for every function $f \in H(\Omega)$, a strict universal sequence $f^{(-n)}$ of n-fold antiderivatives of f is constructed. First we observe that in this construction subsequences of a sequence $\{f^{(-n_k)}\}_{k\in\mathbb{N}}$ realize all the desired approximations. We may choose m_k great enough so as to obtain that the corresponding index n_k belongs to the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$. Then subsequences of the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ are going to be sufficient for the approximation. □

Lemma 2.2. *The following holds*

$$V(\Omega, \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}) = \bigcap_{m=1}^{\infty}\bigcap_{\ell=1}^{\infty}\bigcap_{j_1=1}^{\infty}\bigcap_{j_2=1}^{\infty}\bigcap_{j_3=1}^{\infty}\bigcap_{s=1}^{\infty}\left(\bigcup_{n=1}^{\infty} E(m,\ell,j_1,j_2,j_3,s,n)\right).$$

Proof. We set

$$A = \bigcap_{m=1}^{\infty}\bigcap_{\ell=1}^{\infty}\bigcap_{j_1=1}^{\infty}\bigcap_{j_2=1}^{\infty}\bigcap_{j_3=1}^{\infty}\bigcap_{s=1}^{\infty}\left(\bigcup_{n=1}^{\infty} E(m,\ell,j_1,j_2,j_3,s,n)\right).$$

The inclusion $V(\Omega, \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}) \subset A$ is obvious.
Suppose now that $g \in A$. We will prove that $g \in V(\Omega, \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}})$.
It is easy to see that there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of natural numbers such that properties 1–4 of Definition 1.4 are satisfied (see similar proofs in [18,19]).
In view of the previous lemma, we may realize the fifth approximation with a subsequence $\{\lambda'_n\}_{n\in\mathbb{N}}$ of $\{\lambda_n\}_{n\in\mathbb{N}}$ and the result follows. □

Lemma 2.3. *For every* $m, \ell, j_1, j_2, j_3, s$ *and* $n \in \mathbb{N}$ *the set* $E(m,\ell,j_1,j_2,j_3,s,n)$ *is open in* $H(\Omega)$ *in the topology of uniform convergence on compact subsets of* Ω.

Proof. The set $E(m,\ell,j_1,j_2,j_3,s,n)$ is the intersection of sets which have been proved open (or are obviously open) in $H(\Omega)$ thus it is open (see [20,18,7,4]). □

Lemma 2.4. *The set* $\bigcup_{n=1}^{\infty} E(m,\ell,j_1,j_2,j_3,s,n)$ *is dense in* $H(\Omega)$ *for every* m, ℓ, j_1, j_2, j_3 *and* $s \in \mathbb{N}$.

Proof. Let $f \in H(\Omega)$, $L \subset \Omega$ compact and $\varepsilon > 0$. We will find a function $g \in \bigcup_{n=1}^{\infty} E(m,\ell,j_1,j_2,j_3,s,n)$ such that $\sup_{z\in L} |f(z) - g(z)| < \varepsilon$.
Let $k > m$ be a natural number such that: $L \subset L_k$.

There exist $n_1 \in \mathbb{N}$ and $r > 0$ such that

$$a_n D_m + b_n \subset \Omega,$$

$$a_n D_m + b_n \subset D(b, r) \tag{1}$$

and

$$(L_k \cup K_m) \cap (a_n D_m + b_n) = \emptyset \tag{2}$$

for every $n \geqslant n_1$.
Furthermore, we set

$$M_1 = L_k \cup \overline{D(b, r)} \cup K_m \tag{3}$$

and

$$T = \sup_{z \in M_1} |z| \,. \tag{4}$$

Thus if $f_{j_3}(z) = \sum_{k=0}^{N} c_k (z - \zeta_0)^k$ where $\zeta_0 \in L_m^o$ is fixed, we set

$$p_n(z) = \sum_{k=0}^{N} \frac{c_k (z - \zeta_0)^{k+n}}{(k+1) \cdots (k+n)}$$

and the following holds:

$$p_n^{(n)}(z) = f_{j_3}(z) \,. \tag{5}$$

Observe that f_{j_3} is to be approximated by a derivative, thereby (5) serves this purpose. Next we present some properties of p_n. In particular

$$\sup_{z \in M_1} |p_n(z)| \leqslant \frac{(T + |\zeta_0|)^n}{n!} \sum_{k=0}^{N} |c_k| (T + |\zeta_0|)^k \quad \text{(see (3), (4))} \,.$$

Thus

$$\sup_{z \in M_1} |p_n(z)| \longrightarrow 0 \qquad \text{as } n \longrightarrow +\infty \,. \tag{6}$$

If $R = 4T > 0$ and M_2 is a compact set such that $\bigcup_{\zeta \in L_m} D(\zeta, R) \subset M_2$, we have the following:

$$\begin{aligned} \sup_{\zeta \in L_m} \sup_{z \in M_1} |S_n(p_n, \zeta)(z)| &= \sup_{\zeta \in L_m} \sup_{z \in M_1} \left| \sum_{k=0}^{n} \frac{p_n^{(k)}(\zeta)}{k!} (z - \zeta)^k \right| \\ &\leqslant \sup_{z \in L_m} \sup_{z \in M_1} \sum_{k=0}^{n} \frac{\sup_{w \in D(\zeta, R)} |p_n(w)|}{R^k} |z - \zeta|^k \\ &\leqslant \sup_{w \in M_2} |p_n(w)| \sum_{k=0}^{n} \left(\frac{2T}{R} \right)^k \leqslant 2 \sup_{w \in M_2} |p_n(w)| \,. \end{aligned}$$

One can easily conclude (see the proof of (6)) that $\sup_{w\in M_2}|p_n(w)| \overset{n\to\infty}{\longrightarrow} 0$.
Therefore

$$\sup_{\zeta\in L_m}\sup_{z\in M_1}|S_n(p_n,\zeta)(z)| \longrightarrow 0 \qquad \text{as } n \longrightarrow +\infty\,. \tag{7}$$

Since L_k, K_m are disjoint compact sets with connected complements, we may apply Mergelyan's theorem on $L_k \cup K_m$ to find a polynomial p such that

$$\sup_{\zeta\in L_k}|f(z)-p(z)| < \frac{\varepsilon}{2} \tag{8}$$

and

$$\sup_{z\in K_m}|f_{j_1}(z)-p(z)| < \frac{1}{2s}. \tag{9}$$

Finally, for each $n \geqslant \max\{n_1, \deg p\}$ we use Mergelyan's theorem again, on the compact set $(a_n D_m + b_n) \cup L_k \cup K_m$, to find a polynomial q_n such that

$$\sup_{z\in a_n D_m+b_n}\left|p_n(z)+p(z)-f_{j_2}\left(\frac{z-b_n}{a_n}\right)-q_n(z)\right| \longrightarrow 0 \qquad \text{as } n \longrightarrow +\infty \tag{10}$$

(so that we approximate f_{j_2}; recall that $a_n \neq 0$ for every $n \in \mathbb{N}$) and

$$\sup_{z\in L_k\cup K_m}|q_n(z)|\max\left\{\sum_{k=0}^{n}\left(\frac{2T}{\rho}\right)^k, \frac{n!}{\rho^n}\right\} \longrightarrow 0 \qquad \text{as } n \longrightarrow +\infty, \tag{11}$$

where

$$\rho = \frac{\operatorname{dist}(L_m, \partial L_{m+1})}{2}\,.$$

Hence, for $n \geqslant \max\{n_1, \deg p, \deg f_{j_3}\}$ we set

$$g_n(z) = p_n(z) + p(z) - q_n(z)\,.$$

We shall prove that for some large n, the polynomial g_n is a suitable function to serve our purposes (i.e., g_n satisfies the four properties in order to belong to the set $E(m, \ell, j_1, j_2, j_3, s, n)$ and approximates the function f).
More specifically for the first property of the definition of the set
$E(m, \ell, j_1, j_2, j_3, s, n)$ we have

$$\begin{aligned}
&\sup_{\zeta\in L_m}\sup_{z\in L_m}|S_n(g_n,\zeta)(z)-g_n(z)| \\
&\quad\leqslant \sup_{\zeta\in L_m}\sup_{z\in L_m}|S_n(p_n,\zeta)(z)| + \sup_{z\in L_m}|p_n(z)| \\
&\qquad + \sup_{\zeta\in L_m}\sup_{z\in L_m}|S_n(q_n,\zeta)(z)| + \sup_{z\in L_m}|q_n(z)|\,.
\end{aligned}$$

Using Cauchy estimates we obtain

$$\sup_{\zeta\in L_m}\sup_{z\in L_m\cup K_m}|S_n(q_n,\zeta)(z)| = \sup_{\zeta\in L_m}\sup_{z\in L_m\cup K_m}\left|\sum_{k=0}^{n}\frac{q_n^{(k)}(\zeta)}{k!}(z-\zeta)^k\right|$$
$$\leqslant \sup_{\zeta\in L_m}\sup_{z\in L_m\cup K_m}\sum_{k=0}^{n}\frac{\sup_{w\in\overline{D(\zeta,\rho)}}|q_n(w)|}{\rho^k}|z-\zeta|^k$$
$$\leqslant \sup_{z\in L_{m+1}}|q_n(z)|\sum_{k=1}^{n}\left(\frac{2T}{\rho}\right)^k \qquad \text{(recall (4))} .$$

Since the last expression tends to 0, as $n\to+\infty$ (see (11)) and due to (7) and (6) we obtain

$$\sup_{\zeta\in L_m}\sup_{z\in L_m}|S_n(g_n,\zeta)(z)-g_n(z)|\longrightarrow 0 \qquad \text{as } n\to+\infty . \tag{12}$$

For the second property of g we have:

$$\sup_{\zeta\in L_m}\sup_{z\in K_m}|S_n(g_n,\zeta)(z)-f_{j_1}(z)|$$
$$\leqslant \sup_{\zeta\in L_m}\sup_{z\in K_m}|S_n(p_n,\zeta)(z)| + \sup_{\zeta\in L_m}\sup_{z\in K_m}|S_n(q_n,\zeta)(z)|$$
$$+\sup_{z\in K_m}|f_{j_1}(z)-p(z)| . \tag{13}$$

Combining previous result with relations (7) and (9) we have

$$\sup_{\zeta\in L_m}\sup_{z\in K_m}|S_n(g_n,\zeta)(z)-f_{j_1}(z)| < \frac{1}{2s}+o(1) . \tag{14}$$

Relation (10) is enough to verify the first inequality of property 3.
Now for the fourth property we have
$p_n^{(n)}(z)=f_{j_2}(z)$ and $g_n(z)=p_n(z)+p(z)-q_n(z)$ it follows that

$$\sup_{\zeta\in L_m}|g_n^{(n)}(z)-f_{j_2}(z)| = \sup_{z\in L_m}|q_n^{(n)}(z)|\leqslant \sup_{z\in L_{m+1}}\frac{|q_n^{(n)}(z)|}{\rho^n}n!\overset{n\to\infty}{\longrightarrow}0$$

(see also (11)).
Obviously (see relations (7) and (8)):

$$\sup_{z\in L}|f(z)-g_n(z)| < \frac{\varepsilon}{2}+o(1).$$

Now since the sequence $\{b_n\}_{n\in\mathbb{N}}$ is dense in $\partial\Omega$ there exists a subsequence $\{b_{n_k}\}_{k\in\mathbb{N}}$ which converges to $\zeta^{(\ell)}$. Thus we may choose n_k large enough such that the function g_{n_k} satisfies all requirements. □

Theorem 2.5. *The set $V(\Omega,\{a_n\}_{n\in\mathbb{N}},\{b_n\}_{n\in\mathbb{N}})$ is G_δ and dense in $H(\Omega)$ with the topology of uniform convergence on compacta.*

Proof. Using Lemmas 2.2–2.4 and due to the fact that $H(\Omega)$ is a complete metrizable space, the result follows from Baire's theorem. □

We would also like to give another, equivalent definition of the class $V(\Omega, \{a_n\}, \{b_n\})$.

Definition 2.6. A holomorphic function $f \in H(\Omega)$ belongs to the class $V(\Omega, \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}})$, if for every compact set $K \subset \Omega^c$ with K^c connected, for compact set $M \subset \mathbb{C}$ with M^c connected, for every choice of compact sets $L', \widetilde{L} \subset \Omega$ with connected complements, for every choice of compact sets $L, \widehat{L}, \check{L} \subset \Omega$ and for every choice of continuous functions as follows

$$h : K \longrightarrow \mathbb{C} \qquad h \text{ holomorphic in } K^o,$$
$$\omega : M \longrightarrow \mathbb{C} \qquad \omega \text{ holomorphic in } M^o,$$
$$\psi : L' \longrightarrow \mathbb{C} \qquad \psi \text{ holomorphic in } L'^o,$$
$$\varphi : \widetilde{L} \longrightarrow \mathbb{C} \qquad \varphi \text{ holomorphic in } \widetilde{L}^o,$$

there exist a sequence $\{\lambda_n\}_{n\in N}$ and a strict universal sequence of antiderivatives $f^{(-n)}$ of f such that

1. $\sup_{\zeta\in\widehat{L}} \sup_{z\in\check{L}} |S_{\lambda_n}(f,\zeta)(z) - f(z)| \longrightarrow 0 \quad$ as $n \longrightarrow +\infty$,
2. $\sup_{\zeta\in L} \sup_{z\in K} |S_{\lambda_n}(f,\zeta)(z) - h(z)| \longrightarrow 0 \quad$ as $n \longrightarrow +\infty$,
3. $\sup_{z\in M} |f(a_{\lambda_n} z + b_{\lambda_n}) - \omega(z)| \longrightarrow 0 \quad$ as $n \longrightarrow +\infty$, and $\{b_{\lambda_n}\}_{n\in\mathbb{N}}$ is dense in $\partial\Omega$
4. $\sup_{z\in L'} |f^{(\lambda_n)}(z) - \psi(z)| \longrightarrow 0 \quad$ as $n \longrightarrow +\infty$,
5. $\sup_{z\in\widetilde{L}} |f^{(-\lambda_n)}(z) - \varphi(z)| \longrightarrow 0 \quad$ as $n \longrightarrow +\infty$.

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